Course Review

Chapter 3 - Polynomials and Rational Functions

3-1 Quadratic Functions

Standard Form: $\mathbf{f(x) = ax^2 + bx + c}$; vertex (h,k) --> $h = \frac{-b}{2a}$, k = f(h); y-intercept = c

Vertex Form: $\mathbf{f(x) = a(x-h)^2 + k}$; vertex = (h,k);

Root/Factored Form: $\mathbf{f(x) = a(x - r_1)(x - r_2)}$; roots - r_1, r_2; vertex (h,k) --> $h = \frac{r_1 + r_2}{2}$, k = f(h);

Graphing Quadratic Functions:
- Step 1: find the vertex
- Step 2: find the y-intercept and mirrored point
- Step 3: locate the roots
- Step 4: find other points (and mirrors) as needed
- Step 5: Label at least three points, including the vertex

Writing Quadratic Functions:
- Step 1: decide which form is most appropriate to use
- Step 2: find the values required for the selected form
- Step 3: use parentheses to enter values, and "clean up" carefully
- Step 4: use quadratic regression (calculator) only if necessary

Quadratic Regression:
- Step 1: enter x-data in List 1, and y-data in List 2
- Step 2: plot the data in STAT PLOT
- Step 3: STAT --> CALC --> 5: QuadReg (store in Y-vars)
- Step 4: ZOOM --> 9

3-2 Power Functions: $\mathbf{f(x) = ax^n}$ and $a \neq 0$; $n \neq 0$

(a = constant of variation; n = power)

Even Degree Power Functions:
- symmetric with respect to y-axis
- domain: all real numbers; range: non-negative real numbers
- graph always contains points (0,0), (1,1) and (-1,1)
- as "n" increases: more vertical where $x < -1$ and $x > 1$; flatter and nearer the x-axis where $-1 < x < 1$

Odd Degree Power Functions:
- symmetric with respect to the origin
- domain: all real numbers; range: all real numbers
- graph always contains points (0,0), (1,1) and (-1,-1)
- as "n" increases: more vertical where $x < -1$ and $x > 1$; flatter and nearer the x-axis where $-1 < x < 1$

Power Regression: STAT --> CALC --> A: PwrReg (store in Y-vars)

3-3 Polynomials (sum of power functions)

$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots + dx + e$ *(all exponents are positive integers)*

- If n is EVEN, end behavior is the same; If n is ODD, end behavior is different
- The graph can have n-1 turns (maxima/minima)
- If a > 0, graph RISES to the right; If a < 0, graph FALLS to the right
- The DEGREE of the polynomial is the highest power term
- Polynomials are CONTINUOUS; Domain: $(-\infty,\infty)$
- End behavior is consistent with the the end behavior of power function with same degree as polynomial

Roots and Multiplicity: For a polynomial of degree "n" there are **at most** "n" unique roots.

- Single Root: graph crosses the x-axis (sign changes)
- Double Root: graph touches the x-axis (no sign change)
- Triple Root: graph crosses the x-axis (sign changes) and the root is an inflection point

If you add the multiplicities, you get the degree of the polynomial

Creating Polynomial Functions: Given the roots and any other point, it is possible to write the function

$y = a(x - r_1)(x - r_2)(x-r_3)\ldots(x-r)$;   enter roots, and plug in "other point" to solve for "a"

Factoring and Finding Roots: If one polynomial factors evenly into another, it is a factor of that polynomial.

Long Division

$$\begin{array}{r|l} & x^2 + 2x - 2 \\ \hline 2x+3 & 2x^3 + 7x^2 + 2x + 9 \\ & 2x^3 + 3x^2 \\ \hline & 4x^2 + 2x \\ & 4x^2 + 6x \\ \hline & -4x + 9 \\ & -4x - 6 \\ \hline & 15 \end{array}$$

Synthetic Division

Example: Divide $5x^3 - 8x^2 + 9x + 12$ by $x - 3$.

3	5	-8	9	12
		15	21	90
	5	7	30	102

The quotient is $5x^2 + 7x + 30$ and the remainder is 102.

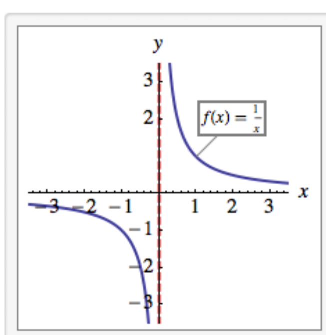
As a result, $\dfrac{5x^3 - 8x^2 + 9x + 12}{x-3} = 5x^2 + 7x + 30 + \dfrac{102}{x-3}$.

3-4 Rational Functions I (ratio of two other polynomials)

$f(x) = \dfrac{P(x)}{Q(x)}$ *(The denominator $\neq 0$)*

Parent Function:

$f(x) = \dfrac{1}{x}$



Domain: $x \neq 0$

Range: $y \neq 0$

Discontinuties: x = 0

Horizontal Asymptote: x = 0

Vertical Asymptote: y = 0

Shape: hyperbola

Proper Form (of a rational function): degree of numerator is less than the degree of the denominator

proper: $\frac{x}{x^2+3}$ improper: $\frac{x^2-1}{x+1}$ *(Use long division to put into proper form if necessary)*

Discontinuities: where the domain does not exist because Q(x) = 0. Two types:

- Holes: points where domain does not exist (canceled out by factoring)
- Vertical Asymptotes: vertical lines that a curve approaches as it heads toward infinity

3-5 Rational Functions II

Horizontal/Slant Asymptotes: Compare the degree of P(x) and Q(x)

- If **P(x) < Q(x)**: horizontal asymptote at y = 0
- If **P(x) = Q(x)**: horizontal asymptote at $\frac{aP(x)}{aQ(x)}$ *(ratio of coefficients of P(x) over Q(x))*
- If **P(x) > Q(x)** ***by exactly one degree***: slant asymptote whose equation is found by long division

Graphing Rational Functions:

Step 1: Record any discontinuities
Step 2: Factor both numerator and denominator
Step 3: Cancel any common factors, recording holes
Step 4: Find roots of denominator to determine any vertical asymptotes
Step 5: Check P(x) and Q(x) and find horizontal/slant asymptotes
Step 6: Find roots of numerator to determine any x-intercepts
Step 7: Set x = 0 and find y-intercept
Step 8: Use sign analysis as needed to determine placement of curves

Writing Rational Functions:

$$\mathbf{f(x) = \frac{a(x-r_1)(x-r_2)(x-hole_1)(x-hole_2)}{(x-VA_1)(x-VA_2)(x-hole_1)(x-hole_2)}}$$

(NOTE: horizontal/slant asymptote will determine "a")

3-6 Polynomial Inequalities

Algebraic Solution:

Step 1: Write inequality so function is on left and zero is on right
Step 2: Find all numbers which cause function to be zero or (if rational) undefined
Step 3: Use the number from step 2 to separate the real number line into intervals
Step 4: Select a test number in each interval and evaluate the function at these test numbers

- If f(x) > 0 then all numbers in the interval are positive
- If f(x) < 0 then all numbers in the interval are negative

Graphical Solution: graph both functions on the same screen and look for intersections. Determine relative position of the functions between these intersections.

3-7 Real Zeros of a Polynomial Function

Remainder Theorem: If f(x) is divided by (x-c), then the remainder is f(c)

EX: x^3-4x^2-5 divided by x-3: Find f(3) = -14 --> therefore $\frac{x^3-4x^2-5}{x-3}$ would have reminder $\frac{-14}{x-3}$

Factor Theorem: (x-c) is a factor of f(x) if and only if f(c) = 0

EX: f(x) = 2x^2-x^2+2x-3 --> Is x-1 a factor? --> f(1) = 0 --> therefore YES

Number of Real Zeros: a polynomial of degree "n" has at most n real zeros.

Rational Roots Theorem: Given a polynomial **a_n +.......+ a_0**,
where q = all factors of a_n and p = all factors of a_0, then p/q (in lowest terms) is a rational zero of f(x)

EX: $2x^3 + 8x^2 - 3x - 2$ p: (factors of -4) = $\pm 1, \pm 2, \pm 4$
q: (factors of 2) = $\pm 1 \pm 2$

All possible factors of p/q: $\pm\frac{1}{2}$, ± 1, ± 2, ± 4

(NOTE: We could now try them all using Factor Theorem, but this is inefficient.)

Decartes Rule of Signs: Tells how many positive and negative real zeros a polynomial has
Count the **variation in sign** throughout the polynomial.

EX: $f(x) = x^5 - 2x^3 + 2x^2 - 3x + 12$ --> There are FOUR sign changes.

- Positive Real Roots: The number of positive real roots is either the number of variation of f(x) or less than that by an even number (EX: 4, 2, or 0 positive real roots)
- Negative Real Roots: The number of negative real roots is either the number of variation of f(-x) or less than that by an even number (EX: $f(-x) = -x^5 + 2x^3 + 2x^2 + 3x + 12$ --> 1 negative real root)

NOTE: Since f(x) is 5th degree, and complex roots appear in pairs, the options for the polynomial are:

Positive Real Roots	Negative Real Roots	Complex Roots
4	1	0
2	1	2
0	1	4

Steps for Finding the Real Zeros of a Polynomial Function

- Step 1: Use degree of polynomial to determine the maximum number of zeros, and Descartes Rule of Signs to determine how many positive, negative, and complex zeros.
- Step 2: If the polynomial has real integer coefficients, use the Rational Zeros Theorem to identify potential zeros
- Step 3: Graph the polynomial and look for possible clues about roots

- Step 4: Use substitution, synthetic division, or long division to test a potential rational zero.
- Step 5: Each time a zero is found (ie, a factor), repeat Step 4 on the "depressed equation".

Bounds on Zeros: A bound (M) on the zeros of f(x) is the smaller of two numbers, if the lead coefficient = 1

$\text{MAX}\{1, |a_0| + |a_1| + \ldots + |a_{n-1}|\}$ ← one OR the sum of the absolute values of lead coefficients, not including a_n

$1 + \text{MAX}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}$ ← one PLUS the largest coefficient absolute value, not including a_n

EX: Find the bounds on zeros of: $f(x) = x^5 + 3x^3 - 9x^2 + 5$

$\text{MAX}\{1, |5| + |-9| + |3|\} = \text{Max}\{1, 17\} = 17$

$1 + \text{MAX}\{|5|, |-9|, |3|\} = 1 + 9 = 10$ ← Since 10 is smaller than 17, Bounds: -10, 10

Intermediate Value Theorem: If f(x) is a continuous function, f(a) > f(b), and f(a) and f(b) are of opposite signs, then f(x) has at least one zero between a and b.

3-8 Complex Roots / Fundamental Theorem of Algebra

Fundamental Theorem of Algebra: Every complex polynomial function f(x) of degree n≥1 has at least one complex zero, and can be factored into "n" linear factors (not necessarily distinct) of the form:

$\mathbf{f(x) = a_n(x-r_1)(x-r_2)\ldots(x-r_n)}$ where $r_1, r_2, \ldots, r_n$ are complex numbers of the form **a + bi**

Conjugate Pairs Theorem: If r = a + bi is a zero of the function f(x), then the complex conjugate (r = a – bi) is also a root of f(x). (For polynomials with coefficients that are real numbers, the zeros occur in conjugate pairs.)

- ➔ A polynomial f(x) of ODD degree MUST therefore have at least one REAL zero.

  EX: A polynomial of degree 5, whose coefficients are real numbers, has the known zeros 1, 5i, and 1 + i. Find the remaining two zeros. Solution: -5i and 1 - i

NOTE: Every polynomial with real coefficients can be uniquely factored into the product of linear Factors (degree = 1) and/or irreducible quadratic factors (degree = 2).