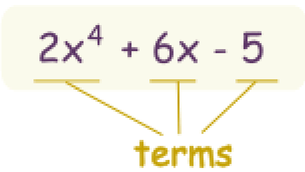
Strategies for Factoring Polynomials

Polynomials

A polynomial is an expression of more than two algebraic terms, especially the sum of several terms that contain different powers of the same variable(s).

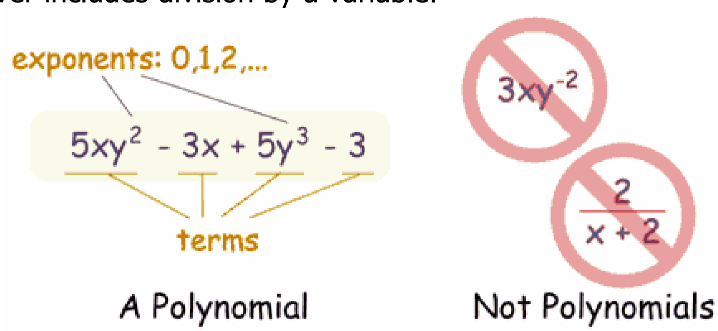
Polynomials can have constants, variables and exponents, but:

* no division by a variable.
* a variable's exponents can only be 0,1,2,3,... etc.
* it can't have an infinite number of terms.



Example of a polynomial:
- this polynomial has 3 terms so it is called a trinomial
- this polynomial has the highest power of 4 so it is 4th degree

Remember, a polynomial never includes division by a variable.



Factoring

"Prime Factorization" is finding which prime numbers multiply together to make the original number.

Example:

32
2 x 16
2 x 2 x 8
2 x 2 x 2 x 4
2 x 2 x 2 x 2 x 2

Factoring Polynomials?

Factoring polynomials is the same as factoring any number. We are trying to break the number down into its lowest (prime) factors.

Greatest Common Factor

For a polynomial, no matter how many terms it has, always check for a greatest common factor (GCF) first. Literally, the greatest common factor is the biggest expression that will go into all of the terms. Using the GCF is like doing the distributive property backward.

Example:

$2x^3 + 6x^2 - 8x$

All terms have a greatest common factor of **2x.**

Put the GCF out front, and then divide the original polynomial by the GCF to find the remaining factor:

$2x(x^2 + 3x - 4)$

Use the distributive property to check your work. If you end up with the original polynomial, you factored correctly.

Special Polynomial Identities

Square Trinomials

$(a + b)^2$ and $(a - b)^2$

$(a + b)^2 = a^2 + 2ab + b^2$

Example: $(x + 3)^2$ $a = x$; $b = 3$ --> $x^2 + (2)(3)(x) + 3^2 = x^2 + 6x + 9$

$(a - b)^2 = a^2 - 2ab + b^2$

Example: $(3x - 2)^2$ $a = 3x$; $b = 2$ --> $(3x)^2 - (2)(3x)(2) + 2^2 = 9x^2 - 12x + 4$

Difference of Two Squares

$a^2 - b^2 = (a + b)(a - b)$

Example: $x^2 - 16$

1) Find the **a** and **b** values: $a = x$; $b = 4$ (the square roots)
2) Plug in the values $(a + b)(a - b)$: $(x + 4)(x - 4)$
3) Use the FOIL method to check your work.

Difference of Two Cubes

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example: $8x^3 - 64$

1) Find the **a** and **b** values: $a = 2x$; $b = 4$ (the cube roots)

2) Plug in the values $(a - b)(a^2 + ab + b^2)$: $(2x - 4)(4x^2 + 8x + 16)$:
3) Use the distributive property to check your work

<u>Sum of Two Cubes</u>

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Example: $x^3 + 27$

1) Find the **a** and **b** values: $a = x$; $b = 3$ (the cube roots)
2) Plug in the values $(a - b)(a^2 - ab + b^2)$: $(x + 3)(x^2 - 3x + 9)$:
3) Use the distributive property to check your work

Factoring Trinomials

Example:

$x^2 + 3x - 4$

The expression $(x^2 + 3x - 4)$ has three terms, and is called a *trinomial*.

1) Draw parenthesis for your factors: (   )(   )
2) Determine the signs for each factor:
   - If **c** is positive, then the factors you're looking for are either both positive or else both negative.
     - If **b** is positive, then the factors are positive
     - If **b** is negative, then the factors are negative.
   - If **c** is negative, one factor is positive and one factor is negative.
     - If **b** is positive, then the larger factors is positive
     - If **b** is negative, then the larger factors is negative.
3) Determine factors of **c** that have a sum of **b**: $(4)(-1) = -4$; $-1+4 = 3$
5) Fill in the parenthesis: $(x + 4)(x - 1)$
6) You may have to try more than one set of factors and check your answer if the solution is not obvious!
7) Use the FOIL method to check your work

Factoring Trinomials (with grouping) - Leading coefficient not 1

Example:

$12x^2 - 17x - 5$

1) Multiply the coefficients from the **a** and **c** terms: $(12)(-5) = -60$
2) Find the factors of -60 that have a sum of -17: -20 and 3
3) Rewrite the expression: $12x^2 + 3x - 20x -5$
4) Group the terms: $(12x^2 + 3x) - (20x -5)$
5) Factor out the GCF for each group: $3x(4x + 1) - 5(4x + 1)$
6) Factor out the common factor: $(4x + 1)(3x - 5)$

Factoring Higher Degrees - "in pairs" (with grouping)

Example:

$x^3 + 2x^2 + 4x + 8$

1) Group the terms in pairs: $(x^3 + 2x^2) + (4x + 8)$
5) Factor out the GCF for each group: $x^2(x + 2) + 4(x + 2)$
6) Factor out the common factor $(x + 2)$: $x^2(x + 2) + 4(x + 2) = (x + 2)(x^2 + 4)$

Solving Equations by Factoring

A quadratic equation is one in which the highest power of x (the degree of the equation) is two. That predicts that it will have two solutions. Similarly, a degree three equation (example: $4x^3 – 6x^2 + 2x – 1 = 0$) will have three solutions.

If an equation can be factored, it can easily be solved:

Step 1: Get all terms on the left side of the equation with 0 remaining on the right side.
Step 2: Factor
Step 3: Set each factor involving x equal to 0 and solve for x.

The solutions are also called <u>roots</u> or <u>zeros</u>.

Example:

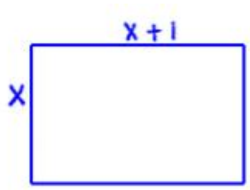
$x^2 = 3x + 10$

Step 1: $x^2 - 3x - 10 = 0$
Step 2: $(x + 2)(x - 5) = 0$
Step 3: $x + 2 = 0 \rightarrow x = -2$; $x - 5 = 0 \rightarrow x = 5$

Solving Word Problems with Factoring

When solving be aware of meaningless answers. For example, when solving for the length of a rectangle, a negative answer would be meaningless.

Example: A rectangle is one foot longer than it is wide. Its area is 20. What are the dimensions?



Length = $x+1 = 4+1 = \boxed{5}$
Width = $x = \boxed{4}$

$x(x+1) = 20$
$x^2 + 1x - 20 = 0$
$(x+5)(x-4) = 0$

$x+5=0$ $x-4=0$
$x = \cancel{-5}$ reject $x = 4$

Binomial Expansion

How do we raise the binomials (a + b) to higher powers?

$(a + b)^0 = 1$
$(a + b)^1 = a + b$
$(a + b)^2 = a^2 + 2ab + b^2$

One way to obtain these higher powers is with repeated multiplication.

$(a + b)^3 = (a + b)\ (a + b)\ (a + b) = (a^2 + 2ab + b^2)\ (a + b) = a^3 + 3a^2b + 3ab^2 + b^3$

This process does work but is very tedious and time consuming, especially for the higher powers.

<u>Binomial expansion is a shortcut!</u>

As an example, consider $(a + b)^5$

Step 1: Obtain the powers of a by starting at the highest power (5) and stepping down to zero (0):

$(a + b)^5 = ?\mathbf{a^5}b^? + ?\mathbf{a^4}b^? + ?\mathbf{a^3}b^? + ?\mathbf{a^2}b^? + ?\mathbf{a^1}b^? + ?\mathbf{a^0}b^?$

Step 2: Obtain the powers of b by starting at zero (0) and stepping up to the highest power (5):

$(a + b)^5 = ?a^5\mathbf{b^0} + ?a^4\mathbf{b^1} + ?a^3\mathbf{b^2} + ?a^2\mathbf{b^3} + ?a^1\mathbf{b^4} + ?a^0\mathbf{b^5}$

Step 3: Obtain the coefficients from Pascal's Triangle:

Pascal's triangle:

1............................ for $(a + b)^0$
1 1........................ for $(a + b)^1$
1 2 1.................... For $(a + b)^2$
1 3 3 1................ for $(a + b)^3$
1 4 6 4 1........... for $(a + b)^4$
1 5 10 10 5 1....... for $(a + b)^5$
1 6 15 20 15 6 1.... for $(a + b)^6$

- The top and outside numbers are all 1.
- Each number in the interior is the sum of the two nearest numbers in the row directly above it.

$(a + b)^5 = \mathbf{1}a^5b^0 + \mathbf{5}a^4b^1 + \mathbf{10}a^3b^2 + \mathbf{10}a^2b^3 + \mathbf{5}a^1b^4 + \mathbf{1}a^0b^5$

Solution: $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Simplify

Keep going until you cannot factor any more!

Strategy

1) Check for a Greatest Common Factor
2) Look for a Special Polynomial Identity
3) Basic Factoring:

    Trinomial

        Leading coefficient = 1
        Leading coefficient > 1

    Higher Degree

        Break into pairs

    Binomial

        Binomial Expansion

4) If solving an equation

    a) Rearrange, if necessary, to make polynomial equal to zero
    b) Find all solutions (called "zeros" or "roots")
    c) Reject any answers that do not fit the physical constraints of the problem